

Radiation from Excited Vortex in the Abelian Higgs Model*

by

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Abstract

Excitation of a vortex in the Abelian Higgs model is investigated with the help of a polynomial approximation. The excitation can be regarded as a longitudinal component of the vector field trapped by the vortex. The energy and profile of the excitation are found. Back-reaction of the excitation on the vortex is calculated in the small κ limit. It turns out that in the presence of the excitation the vortex effectively becomes much wider - its radius oscillates in time and for all times it is not smaller than the radius of the unexcited vortex. Moreover, we find that the vector field of the excited vortex has long range radiative component. Bound on the amplitude of the excitation is also found.

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1 Introduction

There are many reasons for detailed studies of dynamics of vortices. Let us mention here the cosmic string hypothesis [1], condensed matter physics [2], QCD flux-tube [3] and a search for a physical model of a relativistic string [4].

Cumulative efforts by various groups of physicists have unravelled many aspects of fascinating dynamics of vortices. Vortex is a quite complex object – due to its macroscopic thread-like extension it has nontrivial time-evolution with occasional self-interactions of various kinds like scattering [5] or interconnection [6]. Moreover, because of its finite transverse size it can also act as effective potential well trapping various fields. The first example of this kind was the superconducting string [7]. It turns out that this can happen also in the more restrictive framework of Abelian Higgs model [8, 9]. Here the trapped fields are just some components of the vector field. Because other components of the same vector field form the vortex itself, it is natural to regard this phenomenon as the excitation of the vortex. This aspect of dynamics of the vortex becomes important when the vortex is subjected to such violent transitions like the interconnection or hard scattering on an external potential. In fact, numerical investigations of the interconnection clearly show that after this process the involved parts of vortices are excited, [10]. The vortex can also become excited by interaction with particles scattered on it.

Excitations of the vortex in the Abelian Higgs model were studied recently in the paper [11]. In that paper an effective membrane description of the vortex was proposed. The membrane was identified with the surface on which the energy density is maximal - for a straight-linear, static vortex with winding number $|n| \geq 2$ it is a cylinder of finite radius. In the membrane model there exists solution which describes a symmetric bulb travelling along the straight-linear cylinder. It corresponds to a local excitation of the vortex. The shortcoming of such effective model is that it does not yield information on detailed form of the excited vortex fields.

Comprehensive numerical study of various excitations of local and global vortices has been carried out by the authors of Ref.[12]. The excitation we consider in the present paper has not been investigated there.

In general, excitations of the vortex can be divided into two classes, according to the involved components of fields. In the first class there are excitations which do not involve new components of the gauge or Higgs fields – only modes of those components which form the vortex itself are excited. For the vortex these are the modulus of the Higgs field and the A_θ component of the vector field (θ denotes the azimuthal angle). The other class of excitations requires to extend the vortex Ansatz by allowing for non-trivial values of additional components of the Higgs or the vector fields, i.e. the phase of the Higgs field or the A_z component of the vector field (the A_0 component is equivalent to the phase by a gauge transformation). These latter excitations should rather be regarded as bound

states of the additional fields with the vortex. The excitations are usually investigated in a linear approximation – they are treated as small corrections to the basic vortex fields.

The excitation we are interested in belongs to the second class – it involves the A_z component. It can be regarded as a bound state of longitudinal component of the vector field with the vortex. In the paper [9] the existence of the excitation has been proved, and it has been shown that it can be represented by a 2-dimensional Proca field which can propagate along the vortex. These results have been obtained in a linear approximation with respect to deviations of the fields from the static vortex solution. Merely a rough description of the excitation was given – the main obstacle was the lack of analytic expression for the vortex solution even in the Bogomol’nyi limit [13]. Many interesting questions were left without answer, e.g., what are properties of the excitation beyond the linear approximation.

The excitation seems to exist for all values of the parameters of the Abelian Higgs model compatible with the existence of the stable vortex of topological charge $+1$ [9]. Nevertheless, in the present paper we concentrate mainly on the case of very small κ , where it is possible to give explicit and relatively simple analytic formulae. Here $\kappa \equiv m_A/m_H$, where m_A and m_H denote the masses of the vector and Higgs particles, respectively.

We would like to present a detailed study of the excitation in the case $\kappa < 1/2$. Our considerations are based on approximate, analytic expression for the vortex functions obtained in (appropriately adapted) so called polynomial approximation, used till now for domain walls [14]. Next, we obtain formulae for the profile and energy of the excitation. This in turn makes it possible to calculate back-reaction of the excitation on the vortex. The back-reaction is calculated in the case of very small κ when certain simplifications occur. We find that in the presence of the excitation the vortex has oscillating radius, and we calculate the frequency and amplitude of these oscillations. We also notice that there seems to exist an upper bound on amplitude of the excitation. Such a bound reflects the non-linear character of the vortex dynamics and it could not be found in the linear approximation. However, the most striking result is that the excited vortex field has a long range component which is interpreted as radiation of the vector field. Also this phenomenon is absent in the linear approximation.

The general conclusion of our paper is that the effects of the back-reaction are very important – they qualitatively change the picture of the excited vortex obtained in the linear approximation.

The plan of our paper is as follows. In the next Section we present the polynomial approximation for the vortex. In Section 3 the profile and energy of the excitation are calculated. In Section 4 we investigate the back-reaction of the excitation on the vortex. In Section 5 we have collected several remarks about the method used in our paper and about the obtained results.

2 The polynomial approximation for the unexcited vortex

We consider Euler–Lagrange equations of the Abelian Higgs model:

$$(\partial_\nu + iqA_\nu)(\partial^\nu + iqA^\nu)\Phi + \frac{\lambda}{2}\Phi(|\Phi|^2 - \frac{2m^2}{\lambda}) = 0, \quad (1)$$

$$\partial_\mu F^{\mu\nu} = iq(\Phi^* \partial^\nu \Phi - \Phi \partial^\nu \Phi^*) - 2q^2 A^\nu |\Phi|^2, \quad (2)$$

where $m, q, \lambda > 0$. The mass of the Higgs particle is equal to $m_H^2 = 2m^2$, while for the vector field $m_A^2 = \kappa^2 m_H^2$, where $\kappa^2 \equiv 2q^2/\lambda$. Signature of the space-time metric is $(+1, -1, -1, -1)$.

The solution of Eqs. (1),(2), representing an infinite, straight-linear, topological charge +1 vortex lying along the x^3 -axis, is sought with the help of the axially symmetric Ansatz

$$\Phi = \sqrt{\frac{2m^2}{\lambda}} e^{i\theta} F(\rho), \quad (3)$$

$$A_0 = A_3 = 0, \quad (4)$$

$$A^1 = \frac{x^2}{\rho} H(\rho), \quad A^2 = -\frac{x^1}{\rho} H(\rho), \quad (5)$$

where $\rho = \sqrt{(x^1)^2 + (x^2)^2}$ is the radius in the (x^1, x^2) plane and $\theta = \arctan(x^2/x^1)$ is the azimuthal angle. The functions F, H obey the following boundary conditions:

$$F(0) = H(0) = 0, \quad (6)$$

$$F(\infty) = 1, \quad \lim_{\rho \rightarrow \infty} \rho H(\rho) = -\frac{1}{q}. \quad (7)$$

From (1-5) we obtain equations for F and for $\chi \equiv q\rho H(\rho) + 1$:

$$F'' + \frac{F'}{r} - \frac{1}{r^2} \chi^2 F + \frac{1}{2}(F - F^3) = 0, \quad (8)$$

$$\chi'' - \frac{\chi'}{r} - \kappa^2 F^2 \chi = 0, \quad (9)$$

where $r \equiv m_H \rho$, and $'$ denotes d/dr . Boundary conditions for χ have the form

$$\chi(0) = 1, \quad \chi(\infty) = 0. \quad (10)$$

The magnetic field of the vortex is $B^i = \delta^{i3} B(r)$, where

$$B = -\frac{m_H^2}{q} \frac{\chi'}{r}. \quad (11)$$

The main idea of the polynomial approximation is to approximate the fields inside the axially symmetric vortex by simple polynomials in r , and to match the polynomials smoothly with asymptotic form of the vortex fields found for $r > r_0$, where r_0 is the matching point. We shall call r_0 the matching radius of the vortex. Accuracy of such approximation in general increases with the order of the polynomials. In the present Section we shall assume a third order polynomial for F and a sixth order polynomial for χ . Inserting these polynomials in Eqs. (8),(9) we find that the polynomial for F contains only odd powers of r , and the polynomial for χ only even ones. Thus, denoting these polynomials by underlined letters, we have

$$\underline{F} = f_1 r - \frac{1}{3!} f_3 r^3, \quad (12)$$

$$\underline{\chi} = 1 - \frac{1}{2} h_2 r^2 + \frac{1}{4!} h_4 r^4 - \frac{1}{6!} h_6 r^6. \quad (13)$$

The order of the polynomial $\underline{\chi}$ is the maximal one compatible with the assumed third order for the polynomial \underline{F} – in order to compute the term $\sim r^8$ in $\underline{\chi}$ we would have to know the function F in Eq. (9) up to r^5 . Equations (8), (9) give the following recurrence relations for the coefficients of the polynomials

$$\begin{aligned} f_3 &= \frac{3}{4} \left(\frac{1}{2} + h_2 \right) f_1, \\ h_4 &= 3\kappa^2 f_1^2, \\ h_6 &= 5\kappa^2 f_1 (2f_3 + 3h_2 f_1) \end{aligned} \quad (14)$$

The asymptotic values $\tilde{F}, \tilde{\chi}$ of the fields F, χ outside of the vortex, i.e. for $r > r_0$, where r_0 will be determined later on, we find using Eqs.(8),(9) and the boundary conditions (7),(10). There is a complication due to the fact that the asymptotics of the Higgs field depends on κ [17]. For $\kappa < 1/2$ it is governed by the $\chi^2 F/r^2$ term in Eq.(8), while for $\kappa \geq 1/2$ it is governed by the Higgs field mass term. In the following we shall assume that $\kappa < 1/2$. This is mainly to limit the body of our paper – our methods of calculations of the properties of the excitation can be applied for any κ . An additional reason is that for small κ certain simplifications are possible, e.g., one can use a kind of adiabatic approximation.

If for the Higgs field we take

$$\tilde{F}^{(0)} = 1, \quad (15)$$

then in the asymptotic region Eq.(9) is effectively simplified to

$$\tilde{\chi}'' - \frac{\tilde{\chi}'}{r} - \kappa^2 \tilde{\chi} = 0,$$

and its vanishing for large r solution is given by

$$\tilde{\chi}^{(0)} = c_0 r K_1(\kappa r), \quad (16)$$

where K_1 is the modified Hankel function [15] and c_0 is a constant (to be determined later on). In the next step, one could insert the asymptotics (16) for χ into Eq.(8) and to determine corrections to the leading asymptotics (15). Iterating these steps and taking higher order polynomials for \underline{F} and $\underline{\chi}$ we can obtain a very accurate description of the vortex which will be presented in a separate paper [16]. In the present paper, devoted to the excitation rather than to the vortex, we will be satisfied with rather crude description of the unexcited vortex.

The last step in obtaining the static vortex solution is to match the asymptotics with the polynomials (12), (13). We know from [17] that for the exact static vortex solution F is real analytic in (x^1, x^2) . As for the magnetic field, it is also true at least in the Bogomol'nyi limit. For this reason, the number of the matching relations depends only on the number of arbitrary constants present in the asymptotic solutions and in the polynomials - we just require maximal possible smoothness of the fields within the adopted Ansatz (12,13,15,16). In the present paper we will approximate F and χ by functions of r of the C^1 class at $r = r_0$, and real analytic for $r \neq r_0$. More accurate and smoother approximations to the vortex fields will be presented in [16]. Thus, we choose the following matching conditions for F and χ :

$$\underline{F}(r_0-) = \tilde{F}(r_0+), \quad \underline{F}'(r_0-) = \tilde{F}'(r_0+), \quad (17)$$

$$\underline{\chi}(r_0-) = \tilde{\chi}(r_0+), \quad \underline{\chi}'(r_0-) = \tilde{\chi}'(r_0+). \quad (18)$$

Here $g(r_0\pm) \equiv \lim_{r \rightarrow r_0\pm} g(r)$ with $g = F, \chi$. The radius r_0 and the coefficients c_0, f_1, h_2 are to be determined from (17),(18). Conditions (17) with (15) taken into account give

$$f_1 = \frac{3}{2r_0}, \quad f_3 = \frac{3}{r_0^3}. \quad (19)$$

i.e.

$$\underline{F}^{(0)} = \frac{3}{2} \frac{r}{r_0} - \frac{1}{2} \left(\frac{r}{r_0} \right)^3. \quad (20)$$

Next, we calculate h_2, h_4 , and h_6 from formulae (14) – in this way we find

$$\underline{\chi}^{(0)} = 1 + \left(\frac{1}{4} - \frac{4}{3r_0^2} \right) r^2 + \frac{9}{32} \frac{\kappa^2}{r_0^2} r^4 + \frac{3}{16} \frac{\kappa^2}{r_0^2} \left(\frac{1}{8} - \frac{1}{r_0^2} \right) r^6. \quad (21)$$

The final step is to obey the matching conditions (18) for χ . Taking $\tilde{\chi} = \tilde{\chi}^{(0)}$ we find that they are equivalent to the following two equations from which we can determine c_0 and r_0

$$-\frac{1}{3} + \frac{1}{4} \left(1 + \frac{3}{8} \kappa^2 \right) r_0^2 + \frac{3}{128} \kappa^2 r_0^4 = c_0 r_0 K_1(\kappa r_0), \quad (22)$$

$$\frac{8 - \frac{3}{2} r_0^2 - \frac{27}{64} \kappa^2 r_0^4}{1 - \frac{3}{4} \left(1 + \frac{3}{8} \kappa^2 \right) r_0^2 - \frac{9}{128} \kappa^2 r_0^4} = 1 + \kappa r_0 \frac{K_1'(\kappa r_0)}{K_1(\kappa r_0)}. \quad (23)$$

Equation (23) can be easily solved for r_0 by numerical methods for any given κ . For very small κ one can also give analytic formula:

$$r_0^2 \cong \frac{16}{3}(1 + \kappa^2 \ln \kappa) + \mathcal{O}(\kappa^2). \quad (24)$$

(we have used the formula $K_1(z) \cong 1/z$ for $0 < z \ll 1$). Thus, for $\kappa \rightarrow 0$ the Higgs radius of the vortex reaches the fixed value $4/\sqrt{3} \approx 2.31$. Comparison with numerical solutions of Eq.(23) shows that formula (24) gives slightly too big value of r_0 , and that the error is only 6% even for $\kappa = 0.3$. For $\kappa = 0.1$ the error is 1% and it decreases rapidly for smaller κ . In general, the radius decreases with increasing κ , see Fig.1.

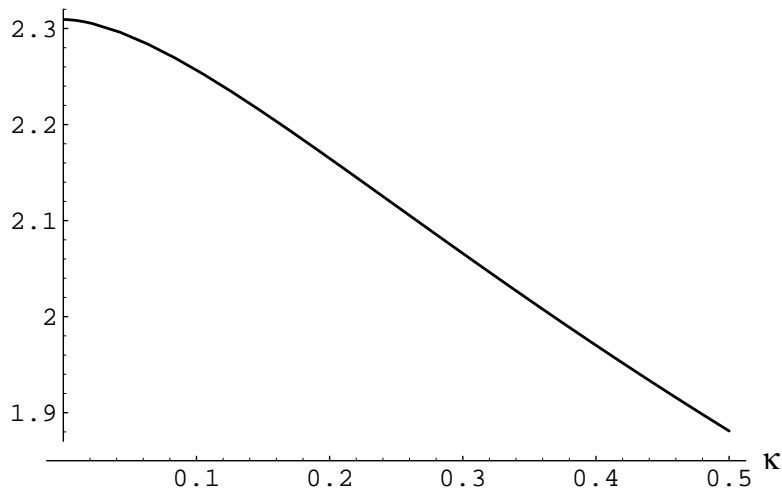


Fig.1. r_0 as a function of κ , determined from Eq.(23).

The constant c_0 is found from Eq(22). In particular, for $\kappa \rightarrow 0$

$$c_0 \cong \kappa(1 - \frac{4}{\sqrt{3}}\kappa^2 \ln \kappa) + \mathcal{O}(\kappa^3). \quad (25)$$

The magnetic field is given on outside of the vortex by formula (11) with $\chi = \tilde{\chi}^{(0)}$ (formula (16)). It has indefinitely increasing range in the limit $\kappa \rightarrow 0$. At the same time its magnitude decreases so that its integral gives the unit of the magnetic flux.

Let us remark that for $\kappa < 1/2$ a more accurate approximation for the asymptotic

field F is provided by the formula

$$\tilde{F} = \sqrt{1 - \frac{2}{r^2} \tilde{\chi}^{(0)2}}.$$

Using this asymptotics instead of $\tilde{F}^{(0)}$ in the matching conditions (17),(18) and a fifth order polynomial for \underline{F} we find that for $\kappa \rightarrow 0$ $r_0 \cong 2.11$. Thus, the improved asymptotics gives slightly smaller matching radius. Such more accurate asymptotics \tilde{F} and a fifth order polynomial \underline{F} are used in Section 4. Yet more accurate asymptotics is given by the formula

$$\tilde{F} = \sqrt{1 - \frac{2}{r^2} \tilde{\chi}^{(0)2} + cK_0(r)},$$

where c is a constant. This asymptotics is considered in [16].

The approximate, analytic form of the vortex functions F, χ we have obtained in this Section for $\kappa < 1/2$ will be used to study the excitation in more detail than it was done in [9].

3 Analytic description of the excitation in linear approximation

The excitation is another vortex-like solution of Eqs. (1,2). It belongs to the same topological class as the vortex itself. Arguments for existence of such a solution, based on a linear approximation, are given in [8],[9]. This solution is time-dependent.

The Ansatz (3-5) is now extended by including $A^3 \equiv A(t, r)$, while $A_0 = 0$ again. The F and χ functions are now time-dependent. In this case equations (1),(2) reduce to the following set of equations

$$-\ddot{F} + F'' + \frac{F'}{r} - \left(\frac{1}{r^2} \chi^2 + \frac{q^2}{m_H^2} A^2 \right) F + \frac{1}{2} (F - F^3) = 0, \quad (26)$$

$$-\ddot{\chi} + \chi'' - \frac{\chi'}{r} - \kappa^2 F^2 \chi = 0, \quad (27)$$

$$-\ddot{A} + A'' + \frac{A'}{r} - \kappa^2 F^2 A = 0, \quad (28)$$

where $t \equiv m_H x^0$ and the dots denote d/dt . Notice that there is no direct coupling of the A field to the χ field, and that A influences F only by the single term $\frac{q^2}{m_H^2} A^2 F$ in Eq.(26).

We shall seek for the solutions of this set of equations using an iterative procedure. First, in the present Section we approximately solve Eq.(28) assuming that the fields F and χ form a background not influenced by the A field, i.e. that they are equal to the initial vortex fields discussed in the previous Section. With the back-reaction of the A field on the vortex switched off, equation (28) becomes linear equation for A with explicitly given

function $F(r)$ – hence the name ”linear approximation”. The back-reaction of the A field on the F and χ fields we shall calculate in the next Section. We hope that if we continued the iterations we would obtain a sequence of approximations convergent towards the exact solution at least for sufficiently small $\frac{g^2}{m_H^2} A^2$ (which is a dimensionless quantity).

Thus, in the present Section we will find the bound-state type solution of the equation

$$-\ddot{A} + A'' + \frac{A'}{r} - \kappa^2 F^{(0)2} A = 0, \quad (29)$$

where $F^{(0)}$ denotes the Higgs field of the unexcited vortex, calculated in the previous Section. We shall use the crude asymptotics (15) and the third order polynomial (12) for the Higgs field – it is sufficient in order to find the excitation and its basic characteristics. More accurate form of the Higgs field would require significantly longer calculations and we think that the improvement would yield merely more accurate numbers, with no change in the overall picture of the excited vortex. If $r \leq r_0$, the functions F , χ and A are again approximated by polynomials \underline{F} , $\underline{\chi}$ and \underline{A} , respectively. For \underline{F} and $\underline{\chi}$ we have formulae (12,13), and

$$\underline{A} = a_0(t) - \frac{1}{2}a_2(t)r^2 + \frac{1}{4!}a_4(t)r^4 - \frac{1}{6!}a_6(t)r^6. \quad (30)$$

We have noted that the coefficients in \underline{A} can be time-dependent. Inserting formula (30) in Eq.(28) we obtain the following set of equations

$$\ddot{a}_0 = -2a_2, \quad (31)$$

$$\ddot{a}_2 = -\frac{4}{3}a_4 + 2\kappa^2 a_0 f_1^2, \quad (32)$$

$$\ddot{a}_4 = -\frac{6}{5}a_6 + 8\kappa^2 a_0 f_1 f_3 + 12\kappa^2 a_2 f_1^2. \quad (33)$$

If $r > r_0$, in Eq.(29) we put $F^{(0)} = 1$. It follows that $A = \tilde{A}$, where

$$\tilde{A} = c_1 \cos(\omega_0 t + \delta) K_0(k_0 r). \quad (34)$$

Here c_1 is a constant, K_0 is the modified Hankel function and

$$\omega_0^2 + k_0^2 = \kappa^2. \quad (35)$$

The frequency ω_0 and the constant c_0 are determined from matching conditions at $r = r_0$:

$$\underline{A}(r_0-) = \tilde{A}(r_0+), \quad (36)$$

$$\underline{A}'(r_0-) = \tilde{A}'(r_0+). \quad (37)$$

We assume that the coefficients a_{2k} , $k = 0, 1, 2, 3$, have the same time-dependence as \tilde{A} , i.e.

$$a_{2k} = \alpha_{2k} \cos(\omega_0 t + \delta)$$

with constant α_{2k} . Then, using Eqs.(31-33), (19) we can express $\alpha_2, \alpha_4, \alpha_6$ by α_0, r_0, κ . We obtain that

$$\underline{A} = \underline{\alpha}(r) \cos(\omega_0 t + \delta), \quad (38)$$

where

$$\underline{\alpha}(r) = \alpha_0 \left[1 - \frac{1}{4} \omega_0^2 r^2 + \frac{1}{64} \left(\omega_0^4 + \frac{9}{r_0^2} \kappa^2 \right) r^4 - \frac{1}{48} \left(\frac{45}{8} \frac{\omega_0^2 \kappa^2}{r_0^2} + 12 \frac{\kappa^2}{r_0^4} + \frac{1}{8} \omega_0^6 \right) r^6 \right]. \quad (39)$$

The matching conditions (36),(37) are equivalent to the following two equations

$$1 - \frac{1}{4} \omega_0^2 r_0^2 + \frac{1}{64} \omega_0^4 r_0^4 - \frac{1}{384} \omega_0^6 r_0^6 - \frac{7}{64} \kappa^2 r_0^2 - \frac{15}{128} \omega_0^2 \kappa^2 r_0^4 = \frac{c_1}{\alpha_0} K_0(k_0 r_0), \quad (40)$$

$$\frac{\omega_0^2 r_0^2 - \frac{1}{8} \omega_0^4 r_0^4 + \frac{1}{32} \omega_0^6 r_0^6 + \frac{45}{32} \omega_0^2 \kappa^2 r_0^4 + \frac{15}{8} \kappa^2 r_0^2}{1 - \frac{1}{4} \omega_0^2 r_0^2 + \frac{1}{64} \omega_0^4 r_0^4 - \frac{1}{384} \omega_0^6 r_0^6 - \frac{7}{64} \kappa^2 r_0^2 - \frac{15}{128} \omega_0^2 \kappa^2 r_0^4} = -2r_0 k_0 \frac{K'_0(k_0 r_0)}{K_0(k_0 r_0)}, \quad (41)$$

where $K'_0(z) \equiv \frac{d}{dz} K_0(z)$ with $z = k_0 r_0$, and r_0 is to be calculated from Eq.(23) of the previous Section.

From Eq.(40) we can determine only the ratio c_1/α_0 , so the amplitude α_0 remains arbitrary in accordance with linearity of Eq.(29) with the fixed $F^{(0)}$.

Equation (41) gives ω_0 as a function of κ . In general, one has to use numerical methods to find this function. It is plotted in Fig.2.

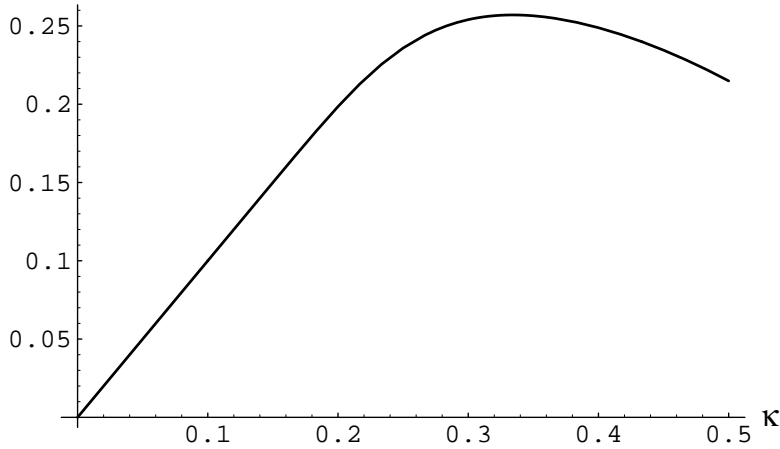


Fig.2. ω_0 as the function of κ , determined from Eqs.(41),(35)

If $\kappa \ll 1$, one can use formula (24) and the following asymptotic formula for the modified Hankel function [15] for $0 < z \ll 1$

$$\frac{zK'_0(z)}{K_0(z)} \cong \frac{1}{\ln(\gamma z) - \ln 2},$$

where $\gamma \approx 1.781072$ is the Euler constant. Formula (35) implies that for $\kappa \rightarrow 0$ ω_0 and k_0 vanish. Keeping only the leading terms on both sides of Eq.(41) we obtain the equation

$$\left(-\frac{23}{16}\kappa^2 + \frac{1}{2}k_0^2\right)r_0^2 \approx \frac{1}{\log k_0},$$

which gives

$$k_0^2 \sim \exp \left\{ -\frac{32}{23} \frac{1}{\kappa^2} \frac{1}{r_0^2} \right\}$$

It is easy to see that the value of the coefficient in front of this formula for k_0 depends on subleading terms in Eq.(41). Taking these terms into account we find that for $\kappa \ll 1$

$$k_0^2 \cong 3.86 \frac{1}{r_0^2} \exp \left\{ -\frac{32}{23} \frac{1}{\kappa^2} \frac{1}{r_0^2} \right\}, \quad (42)$$

where r_0 is given by formula (24). Comparison with numerical solutions of Eq.(41) shows that formula (42) gives slightly too big value of k_0 , with 13% accuracy even for $\kappa = 0.3$. For $\kappa = 0.1$ the error is 2% and it decreases rapidly for smaller κ .

From Eq.(40) we find that for small κ

$$\frac{c_1}{\alpha_0} \cong -\frac{23}{16} r_0^2 \kappa^2. \quad (43)$$

It follows that when $\kappa \rightarrow 0$, the function $c_1 K_0(k_0 r)$ for any finite fixed r tends to a constant value equal to α_0 , hence $\tilde{A} \rightarrow a_0$.

The existence of the excitation in the linear approximation can be seen by regarding the equation

$$-\frac{1}{2} \left(\alpha'' + \frac{\alpha}{r} \right) + \frac{\kappa^2}{2} F^{(0)2} \alpha = \frac{\omega_0^2}{2} \alpha,$$

obtained from Eq. (29) by substituting $A = \cos(\omega_0 t + \delta) \alpha(r)$, as Schrödinger equation for a particle in the 2-dimensional potential well given by $V(r) = \frac{\kappa^2}{2} F^{(0)2}$. It is a well known fact [18] that any potential well on a plane has at least one bound state, hence localized non-zero solution A of Eq.(29). For small κ the potential well has finite width $r_0 \approx 4/\sqrt{3}$ and it is very shallow ($V_{\max} = \kappa^2/2$). Nevertheless, the bound state exists and the corresponding frequency ω_0 , is given by formula (42). Estimate made in [9] shows that the potential well is too weak to support p-wave bound states. Therefore, we do not consider more general, θ -angle dependent A field.

4 The vortex in presence of the excitation

Mathematically, the presence of the excitation changes the vortex because of the A^2 term in Eq.(26). Because $A^2 \sim \cos^2(\omega_0 t + \delta) = \frac{1}{2}[1 + \cos(2\omega_0 t + 2\delta)]$, the perturbation has two components: the static one and the one oscillating with the frequency $2\omega_0$. In order to calculate their influence on the vortex we reanalyse Eqs.(26),(27) assuming that A is given by the approximate solution of Eq.(28) found in the previous Section. We shall consider the limiting case $\kappa \rightarrow 0$ when we can do this analytically.

In the case of very small κ , A^2 changes in time very slowly as the frequency $2\omega_0$ is very small. Because we expect that the induced by the presence of the excitation time dependence of F is also characterized by the frequency $2\omega_0$, the term \ddot{F} on the r.h.s of Eq.(26) is of the order $\omega_0^2 \sim \kappa^2$ and it can be neglected. Therefore, equation (26) approximately reduces to the following equation

$$F'' + \frac{1}{r}F' - \left(\frac{1}{r^2}\chi^2 + \frac{q^2}{m_H^2}A^2\right)F + \frac{1}{2}(F - F^3) = 0. \quad (44)$$

Now let us turn to the Eq.(27) for the field χ . Again, we expect that the induced time dependence of χ is characterized by the frequency $2\omega_0$, so that the term $\ddot{\chi}$ is of the order κ^2 . However, if we neglect this term then we also have to neglect the term $\kappa^2 F^2 \chi$, and Eq.(27) reduces to the following equation

$$\chi'' - \frac{1}{r}\chi' = 0. \quad (45)$$

Its general solution has the form

$$\chi = c_2 + d_2 r^2, \quad (46)$$

where c_2, d_2 are arbitrary constants. It is easy to see that such χ can not obey the boundary conditions (10). Thus, neglecting the terms $\sim \kappa^2$ in Eq.(27) altogether is an oversimplification if we consider the whole range of r , i.e. $[0, \infty)$. On the other hand, it is acceptable approximation if we consider χ in a finite interval of the r variable, e.g., in a vicinity of the matching point, now denoted by r_e . It breaks down only if we try to apply it also for $r \rightarrow \infty$. This means that the limits $\kappa \rightarrow 0$ and $r \rightarrow \infty$ are not interchangeable, and the correct order of them is that first we should find χ for very large r and only after that we can consider the limit $\kappa \rightarrow 0$. One could see this by analysing the vortex solution of Section 2, cf. formulae (16),(25). Later on in this Section we shall find the asymptotics $\tilde{\chi}$ for the very large r . For the current calculations it is sufficient to know that for $r \sim r_e$ χ is given by formula (46).

From Eq.(44) we obtain approximate form of the Higgs field F in the presence of the excitation. As always in the polynomial approximation we separately consider the regions of small and large r . The matching takes place at $r = r_e$. In the region $r \leq r_e$ we will

approximate F by a fifth order polynomial,

$$\underline{F} = f_1 r - \frac{1}{3!} f_3 r^3 + \frac{1}{5!} f_5 r^5. \quad (47)$$

The reason is that the more precise (from that considered in Section 2) form of the asymptotics of the Higgs field which is given below requires at least the fifth order polynomial if the matching conditions (17) are to be satisfied. Instead of the first of recurrence relations (14) we now have

$$f_3 = \frac{3}{4} \left(\frac{1}{2} + h_2 - \frac{q^2}{m_H^2} a_0^2 \right) f_1. \quad (48)$$

We also obtain the relation

$$f_5 = \frac{5}{6} \left(\frac{1}{2} + h_2 - \frac{q^2}{m_H^2} a_0^2 \right) f_3 + \frac{5}{2} \left(\frac{1}{6} h_4 + \frac{1}{2} h_2^2 + f_1^2 - \frac{2q^2}{m_H^2} a_0 a_2 \right) f_1. \quad (49)$$

Notice that passing to the fifth order polynomial does not increase the number of arbitrary constants in our Ansatz, because the new constant f_5 is related to the other constants by formula (49).

For χ inside the vortex we again use the polynomial (13), and now the coefficients h_{2k} can in principle depend on time. However, the corresponding recurrence relations in the limit $\kappa \rightarrow 0$ reduce to

$$h_4 \cong 0, \quad h_6 \cong 0. \quad (50)$$

The matching conditions (18) applied to the polynomial $\underline{\chi}$ and χ given by formula (46) imply that

$$c_2 = 1, \quad d_2 = -\frac{1}{2} h_2. \quad (51)$$

The asymptotics \tilde{F} of the Higgs field obeys the following equation

$$\tilde{F}'' + \frac{1}{r} \tilde{F}' - \left(\frac{1}{r^2} \tilde{\chi}^2 + \frac{q^2}{m_H^2} \tilde{A}^2 \right) \tilde{F} + \frac{1}{2} (\tilde{F} - \tilde{F}^3) = 0, \quad (52)$$

where \tilde{A} is given by formula (34), and $\tilde{\chi}$ is found below. We use the observation following from formulae (34),(43) that for any $r > 0$ in the limit $\kappa \rightarrow 0$

$$(\tilde{A}^2)' \sim \kappa^2.$$

Hence, \tilde{A}^2 changes very slowly with r for small κ . As we shall see below, the same is true for $\tilde{\chi}$, essentially because the mass of this field is equal to κ . Then, it is easy to check that the equation for \tilde{F} has the following approximate solution

$$\tilde{F} = \sqrt{1 - \frac{2}{r^2} \tilde{\chi}^2 - \frac{2q^2}{m_H^2} \tilde{A}^2}. \quad (53)$$

Adopting this solution is equivalent to neglecting the derivative terms ($\sim F', F''$) in Eq.(52).

The next step is to obey the matching conditions (17). Because the matching takes place at finite $r = r_e$ we may use formula (46). We also use formulae (50),(51), and we notice that for finite r and $\kappa \rightarrow 0$

$$\tilde{A}^2 \cong a_0^2,$$

as follows from formulae (34),(42),(43). The matching conditions (17) give the following relation

$$f_1 = \frac{8(5\beta r_e^2 - 12)}{r_e^2(32 - \beta r_e^2)\sqrt{\beta r_e^2 - 2}}, \quad (54)$$

where

$$\beta \equiv 1 - \frac{2q^2 a_0^2}{m_H^2} + 2h_2,$$

as well as the equation

$$x\sqrt{\beta - x}(16x - \beta) = \frac{5\beta - 6x}{\sqrt{\beta - x}} \left(4x^2 - \frac{\beta}{2}x + \frac{\beta^2}{48} + \frac{8}{3} \frac{x^3(5\beta - 6x)^2}{(\beta - x)(16x - \beta)^2} \right) \quad (55)$$

for

$$x \equiv \frac{2}{r_e^2}.$$

Equation (55) has the following solution

$$x = d_0 \beta,$$

where the constant d_0 can be easily determined by numerical methods,

$$d_0 \approx 0.449.$$

Moreover, we shall show later on that $h_2 \approx 0$. Hence,

$$r_e \approx \frac{2.11}{\sqrt{1 - \frac{2q^2}{m_H^2} a_0^2}}. \quad (56)$$

It is clear from this formula and from the formula

$$a_0^2 = \alpha_0^2 \cos^2(\omega_0 t + \delta)$$

that r_e oscillates in the interval $[2.11, 2.11/(1 - 2q^2\alpha_0^2/m_H^2)]$. Let us recall that 2.11 is the matching radius in the absence of the excitation, as mentioned at the end of Section 2.

We see from formula (56) that as α_0^2 approaches $m_H^2/(2q^2)$ the amplitude of oscillations of r_e increases indefinitely, and for still higher amplitude of the excitation formula (56)

gives unphysical r_e . We interpret this result as a hint that there is an upper bound $\sim m_H/(\sqrt{2}q)$ on the amplitude α_0 of the excitation. Such a bound would reflect nonlinear character of the set (26-28) – in the linear approximation no restriction on the amplitude appears. It would be desirable to check the existence of the bound by another method, e.g., by direct numerical analysis of Eqs.(26-28). Interesting question what happens to the vortex if the bound is exceeded we will leave for a separate investigation.

In order to find the $\tilde{\chi}$ field of the excited vortex for the very large r we have to solve equation (27) with $F = \tilde{F}$, where \tilde{F} is given by formula (53). As discussed earlier, now we should not neglect the terms proportional to κ^2 . Inserting formula (53) and neglecting the term $2\kappa^2\tilde{\chi}^3/r^2$ (because we expect that $\tilde{\chi}$ is small for $r > r_e$ in comparison with the other terms – subsequent computation confirms this expectation) we obtain the following equation

$$-\ddot{\tilde{\chi}} + \tilde{\chi}'' - \frac{1}{r}\tilde{\chi}' - m_{eff}^2\tilde{\chi} + \kappa^2\frac{q^2c_1^2}{m_H^2}K_0^2(k_0r)\cos(2\omega_0t + 2\delta)\tilde{\chi} = 0, \quad (57)$$

where we have introduced the notation

$$m_{eff}^2 = \kappa^2(1 - \frac{q^2c_1^2}{m_H^2}K_0^2(k_0r)). \quad (58)$$

Next, we split $\tilde{\chi}$ into a static and time-dependent parts,

$$\tilde{\chi} = \tilde{\chi}_s + \tilde{\chi}_\omega, \quad (59)$$

where the static part by definition obeys the equation

$$\tilde{\chi}_s'' - \frac{1}{r}\tilde{\chi}_s' - m_{eff}^2\tilde{\chi}_s = 0. \quad (60)$$

It follows from formula (59) and Eqs.(57),(60) that $\tilde{\chi}_\omega$ obeys the equation

$$\begin{aligned} -\ddot{\tilde{\chi}_\omega} + \tilde{\chi}_\omega'' - \frac{1}{r}\tilde{\chi}_\omega' - m_{eff}^2\tilde{\chi}_\omega + \kappa^2\frac{q^2c_1^2}{m_H^2}K_0^2(k_0r)\cos(2\omega_0t + 2\delta)\tilde{\chi}_\omega \\ = -\kappa^2\frac{q^2c_1^2}{m_H^2}K_0^2(k_0r)\cos(2\omega_0t + 2\delta)\tilde{\chi}_s. \end{aligned} \quad (61)$$

Thus, $\tilde{\chi}_s$ acts as a source for $\tilde{\chi}_\omega$.

In the present paper we will discuss only a perturbative solution of Eq.(61), obtained by expanding $\tilde{\chi}_\omega$ in powers of the dimensionless amplitude $q\alpha_0/m_H$ of the excitation. This solution is expected to be a good approximation to the exact solution of Eq.(61) if the dimensionless amplitude is small, so from now on we assume that this is the case. The perturbative solution is sufficient to show that the excited vortex radiates the χ field. We start from the observation that the r.h.s. of Eq.(61) together with formula (43) suggest that $\tilde{\chi}_\omega$ is of the order $q^2\alpha_0^2/m_H^2$. Therefore, the last term on the l.h.s. of Eq.(61) is of the

order $(q^2\alpha_0^2/m_H^2)^2$ and can be neglected if the dimensionless amplitude of the excitation is small. For the same reason, m_{eff} in Eq.(61) can be replaced by κ . Solving obtained in this way the simplified version of Eq.(61) we find the lowest order contribution to the perturbative series for $\tilde{\chi}_\omega$.

Let us introduce a new function $h_\omega(r)$

$$\tilde{\chi}_\omega = \frac{\kappa q^2 \alpha_0^2}{m_H^2} \cos(2\omega_0 t + 2\delta) r h_\omega(r). \quad (62)$$

The simplified form of Eq.(61) is equivalent to the following equation for the function h_ω :

$$h_\omega'' + \frac{1}{r} h_\omega' - \frac{1}{r^2} h_\omega + k_1^2 h_\omega = -\frac{\kappa}{r} \frac{c_1^2}{\alpha_0^2} K_0^2(k_0 r) \tilde{\chi}_s(r), \quad (63)$$

where

$$k_1 \equiv \sqrt{4\omega_0^2 - \kappa^2}.$$

Observe that for the very small κ the coefficient k_1^2 is positive because then $\omega_0^2 \sim \kappa^2$, see formulae (35),(42).

The asymptotics $\tilde{\chi}$ is determined by $\tilde{\chi}_s$ and h_ω . Equation (60) has the following vanishing at the infinity, approximate solution

$$\tilde{\chi}_s \approx c_3 \kappa r K_1(m_{eff} r), \quad (64)$$

where c_3 is a constant to be determined later.

General, vanishing at the infinity solution of Eq.(63) is given by the formula

$$h_\omega(r) = c_3 h_\infty(k_1 r) + c_4 J_1(k_1 r) + c_5 N_1(k_1 r), \quad (65)$$

where c_4, c_5 are constants to be determined later on; J_1, N_1 are Bessel and von Neumann functions, respectively [15]; and $h_\infty(z)$ (with $z = k_1 r$) is a particular solution of the inhomogeneous Bessel equation

$$\frac{d^2 h_\infty(z)}{dz^2} + \frac{1}{z} \frac{dh_\infty(z)}{dz} + (1 - \frac{1}{z^2}) h_\infty(z) = -\frac{\kappa^2}{k_1^2} \frac{c_1^2}{\alpha_0^2} K_1(\frac{\kappa}{k_1} z) K_0^2(\frac{k_0}{k_1} z). \quad (66)$$

For small κ we can approximate

$$k_1 \approx \kappa \sqrt{3}.$$

Equation (66) can be solved by a standard method [19],

$$h_\infty(z) = f(z) J_1(z) + g(z) N_1(z), \quad (67)$$

where

$$f(z) = -\frac{\pi c_1^2}{6\alpha_0^2} \int_z^\infty dx x N_1(x) K_1(\frac{x}{\sqrt{3}}) K_0^2(\frac{k_0}{\kappa\sqrt{3}} x), \quad (68)$$

and

$$g(z) = \frac{\pi c_1^2}{6\alpha_0^2} \int_z^\infty dx x J_1(x) K_1\left(\frac{x}{\sqrt{3}}\right) K_0^2\left(\frac{k_0}{\kappa\sqrt{3}}x\right). \quad (69)$$

The constants c_3, c_5 can be fixed by matching the asymptotics $\tilde{\chi}$ with χ , given by formulae (46),(51), at $r = r_e$. This requirement gives the following equations

$$\tilde{\chi}_s(r_e) + \frac{\kappa q^2 \alpha_0^2}{m_H^2} r_e h_\omega(r_e) \cos(2\omega_0 t + 2\delta) = 1 - \frac{h_2}{2} r_e^2, \quad (70)$$

$$\tilde{\chi}'_s(r_e) + \frac{\kappa q^2 \alpha_0^2}{m_H^2} [r_e h'_\omega(r_e) + h_\omega(r_e)] \cos(2\omega_0 t + 2\delta) = -h_2 r_e. \quad (71)$$

Next, we notice that for small κ (when $\kappa r_e \ll 1$)

$$\tilde{\chi}_s(r_e) \approx c_3, \quad \tilde{\chi}'_s(r_e) \approx 0,$$

$$\kappa r_e J_1(\sqrt{3}\kappa r_e) \approx 0, \quad \kappa r_e N_1(\sqrt{3}\kappa r_e) \approx -\frac{2}{\sqrt{3}\pi}.$$

Then it is easy to see that conditions (70),(71) imply that

$$h_2 = 0, \quad c_3 = \sqrt{1 - \frac{q^2 \alpha_0^2}{m_H^2}}, \quad c_5 = -g(0)c_3. \quad (72)$$

The definition (69) gives for $z = 0$

$$g(0) = \frac{\sqrt{3}\pi}{8}. \quad (73)$$

The constant c_4 is not fixed by the matching conditions. The corresponding term in $\tilde{\chi}$ has the following form

$$\delta\tilde{\chi} = c_4 \frac{\kappa q^2 \alpha_0^2}{m_H^2} r J_1(\sqrt{3}\kappa r) \cos(2\omega_0 t + 2\delta).$$

This function, as well as its derivatives, is regular for all r including $r = 0$, and it obeys the wave equation

$$-\delta\ddot{\tilde{\chi}} + \delta\tilde{\chi}'' - \frac{1}{r}\delta\tilde{\chi}' - \kappa^2\delta\tilde{\chi} = 0$$

in the whole space. Therefore, we shall regard $\delta\tilde{\chi}$ merely as an artifact of the linearity of Eq.(57), and interpret $\delta\tilde{\chi}$ as a free, standing electromagnetic wave which is not related to the excited vortex. Because we are interested in the excited vortex alone, we put

$$c_4 = 0. \quad (74)$$

Thus, we have finally obtained that

$$h_\omega(r) = \sqrt{1 - \frac{q^2 \alpha_0^2}{m_H^2}} (h_\infty(\sqrt{3}\kappa r) - \frac{\sqrt{3}\pi}{8} N_1(\sqrt{3}\kappa r)). \quad (75)$$

The function $h_\infty(z)$ vanishes exponentially for large z , but von Neumann function $N_1(z)$ decreases rather slowly for large z ,

$$N_1(z) \cong \sqrt{\frac{2}{\pi z}} \sin(z - \frac{3\pi}{4}).$$

Thus, the $\tilde{\chi}$ field has the long range component equal to

$$-\frac{\sqrt{3}\pi}{8} \frac{q^2 \alpha_0^2}{m_H^2} (1 - \frac{q^2 \alpha_0^2}{m_H^2})^{1/2} \kappa r N_1(\sqrt{3}\kappa r) \cos(2\omega_0 t + 2\delta).$$

Its natural interpretation is that it describes radiation of the vector field from the excited vortex.

It is easy to check that the resulting total field configuration has infinite energy per unit length of the vortex. This is due to the fact that we have considered the excitation of the standing wave type for the straight-linear, infinite vortex in infinite space. The presence of the long range radiative component means that the excited vortex should not be regarded as a localised soliton, eventhough it has the topological charge +1.

5 Remarks

Let us recapitulate results of our paper. First, on a methodological side, we have shown that with the help of the polynomial approximation one can obtain analytic description of the excited vortex, including the profile and frequency of the excitation as well as the back-reaction of the excitation on the vortex. Thus, the polynomial approximation turns out to be a useful tool also for investigations of dynamics of vortices – earlier it has been applied to domain walls [14].

Second, on a physics side, we have shown that the excited vortex contains the radiative component. To find it, one has to calculate the back-reaction of the excitation on the vortex. The back-reaction is due to non-linearity of Eqs.(26-28). In the linear approximation considered in Section 3 the effects which are due to the back-reaction are not taken into account. The radiative component we have found is formed of the vector field. We expect that after several iterations of our procedure, i.e. correcting the vortex fields, then calculating corrections to the excitation field A , next again correcting the vortex fields, etc., one would also obtain a radiative component in the Higgs field. Its amplitude will be proportional to a higher power of the square of the dimensionless amplitude $q^2 \alpha_0^2 / m_H^2$

of the excitation. Taking the back-reaction into account has resulted in drastic alteration of physical characteristics of the excited vortex obtained in the linear approximation.

In the present paper we have considered for simplicity only the standing wave type, x^3 -independent solutions in the infinite space, and we have found that in this case the excitation has infinite energy because of the radiative component. Nevertheless, the obtained solution might be of physical interest. For example, in a real system like a superconductor of finite volume one could sustain the excitation by an external electromagnetic wave and the standing wave regime could be obtained in the whole volume of the superconductor. It is clear that obtaining the standing wave in the infinite volume would require infinite energy.

For description of the excitation created dynamically during, e.g., vortex reconnections, the standing wave, x^3 -independent solution has to be modified. In such an event the excitation is created almost suddenly at certain instant of time with a finite amount of energy and it occupies only a small part of the vortex. It turns out that it is not difficult to appropriately generalize our calculations, so that one can give approximate, analytic description of propagation of such a localised excitation along the vortex, as well as of propagation of radiation wave emitted from the travelling excitation [20]. Solutions of this type have finite energy.

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